On the photon statistics of the fluorescence radiation of a laser-driven single atom

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Abstract. The subjects of the investigation are single atoms, for instance isolated in a Paul trap, which are driven by laser fields while their resonance radiation is measured by photon counters. For a rather wide class of experimental setups or theoretical models, a method is developed to determine the statistics of the photons observed. In contrast to other approaches it avoids referring to 'continuous measurements' or similar concepts, but tries to stay as far as possible within the framework of ordinary quantum mechanics. The method is illustrated by application to simple models.

Introduction

The development of the Paul trap made the spectroscopic investigation of single atomic systems possible. This offers the opportunity for various experiments, similar to that proposed by Dehmelt [1]. They consist essentially of an isolated ion driven by one or several strong laser fields, while the fluorescent light is observed by photon multipliers.

As a model of the internal dynamics, one usually considers an N-level system (atom) coupled to classical, time-dependent fields (laser) and to the quantized radiation field, a procedure that has proved to work well in similar quantum optical investigations. The theoretical description of the observation of the radiation field is commonly based on Glauber's detector theory [2]. Concerning the experiments described above this procedure has been used, for example, in [3] and [4]. Since, however, in this case the state of the radiation field is determined only by the ion, while the ion remains essentially unaffected by the state of the field, another simpler method, which can be expected to be equivalent, is to apply the formal theory of quantum measurements directly on the observables of the radiation field†.

In the present article this second method will be carried out in the following way: one considers a class of (idealized) observations of the radiation field performed by a counter surrounding the atom, which can be opened during arbitrary intervals. These observations are described by means of a quantum mechanical model within the framework of the theory of measurement due to von Neumann and Lüders [5]. It turns out that for all practically possible measurements the results are compatible with a uniquely determined classical counting process in the following sense. The probabilities that the detector gives a signal (or not) during a certain time interval are equal to the

† In comparison to the previous method, this may be regarded as a shift of the 'cut' towards the atom, by which the counters are no longer described dynamically.

probabilities that during this period at least one event (or no event) of the process appears. This suggests that this process can be interpreted as the 'process of photon emission' and also be used for a description of more realistic experiments (which in principle could be treated in the same way as the idealized ones).

Similar approaches have already been followed by other authors, e.g. [6–9], but have always been closely connected to specific models. It is the aim of the present paper to develop this method in as general a way as possible seeking to avoid some of the restrictions and additional assumptions made in the quoted work. For instance, I try to stay, as far as possible, within ordinary quantum mechanics and do not refer, as in [6], to a concept of continuous measurement [10]. Furthermore I want to emphasize that to construct the counting process an interruption of the time evolution by numerous hypothetical measurements, as appears to be suggested by the approach in [8] and similarly in [9], is not necessary. (Indeed it turns out that the results of the measurements are more compatible with the process, the smaller is the number of observations. From the technical point of view the present investigation is partly based on the projector formalism (Nakajima, Zwanzig). This was the case in [7], too; it seems, however, that the formal possibilities of that method have not been exhausted there.

The paper is organized as follows. Section 1 contains a short sketch of the idealized measurements on which the investigation is based. In section 2, the central part of the article, these measurements are described theoretically and conditions for the existence of a compatible classical counting process are discussed. Some more technical items and a discussion of the approximations (Born–Markov) are deferred to section 3. The last section contains simple examples which demonstrate the applicability and indicate the connection of the paper to previous work. In the appendix it is shown that the approach developed in [8], within the framework of its concomitant approximations, yields essentially the same results as the method advocated here.

1. Characterization of the basic experiments

The observations taken as the starting point of the investigations will be idealized but, nevertheless, are assumed to be actually performed and thus chosen to be typical for single-atom experiments. As indicated in the introduction they consist of surrounding the atom by a broad-band 4π-counter being opened at prescribed times t_i, $i = 1, \ldots, n$, and closed again at later times $t_i + \Delta_i < t_{i+1}$. After each shutting one of two possible (macroscopic) results can be registered. Either one 'received a signal' or one 'received no signal'. (The restriction on counters with only two states is for the sake of simplicity only.) The experiments are performed on an ensemble of systems (consisting of atom and field), which is at the initial time t_0 prepared according to a procedure ρ_0 (meaning not yet a density matrix).

The statistics of the results can be expressed by the common probabilities

$$W^{(n)}(t_n \Delta_n \#_n, \ldots, t_1 \Delta_1 \#_1 | t_0 \rho_0) \tag{1}$$

where

$$\# = \begin{cases} 0 & \text{for no signal obtained} \\ > & \text{for signal obtained.} \end{cases}$$

The following investigations aim to derive these common distributions theoretically

from the dynamics of a suitable quantum mechanical model and with the theory of measurement used in the same way as von Neumann and Lüders [5].

2. Theoretical treatment

2.1. The class of models

The dynamical problem considered will be the quantum dynamics of an N-level system (atom (a)) coupled to a quasicontinuous set of oscillators (the modes of the radiation field (b)). The Hamilton operator is assumed to have the form

$$H(t) = H_a(t) + H_b + H_{ab} \tag{2}$$

where

$$H_a(t) = H_a + V_a(t) \tag{2a}$$

describes the free atom and the influence of the laser fields,

$$H_b = \sum_n \omega_n a_n^+ a_n \tag{2b}$$

the free radiation field (n may be a multi-index, i.e. (k, λ)) and

$$H_{ab} = \sum_n (\kappa_n C_n a_n^+ + \kappa_n^* C_n^+ a_n) \tag{2ab}$$

the coupling (dipole approximation). C_n are operators acting on the space of the atom.
   Thus, an ensemble ρ prepared at time t' will develop, up to time t, into

$$T(t, t')\rho \qquad (= U(t, t')\rho U(t, t')^+) \tag{3}$$

with

$$\frac{d}{dt} T(t, t')\rho = -i[H(t), T(t, t')\rho] \qquad T(t, t)\rho = \rho. \tag{4}$$

(Superoperators, i.e. operators acting on Hilbert-space operators, will be denoted by bold letters in the following.)

2.2. On the modelling of the observations

The actions being performed during the measuring procedure at the times

$$t_0, t_1, t_1 + \Delta_1, t_2, t_2 + \Delta_2, \ldots, t_n, t_n + \Delta_n$$

will be described in the frame of the model by the following operations and formal measurements.
   (i) At the initial time t_0 the ensemble considered is assumed to be described by a density matrix $\rho_{a0} \times |v\rangle\langle v|$ ($|v\rangle$ is the vacuum of the radiation field).
   (ii) At times t_i the radiation field is set back instantaneously into the vacuum state†. If immediately before such an operation the state was ρ, so it is shortly afterwards $(\mathrm{Tr}_b \rho) \times |v\rangle\langle v|$.

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† To motivate this operation, one may think of the counter being substituted by one filling the hole space up to a near neighbourhood of the atom, and being 'free of field excitations' before the shutter is opened.

(iii) At the times $t_i + \Delta_i$ formal measurements are performed on the radiation field, which are defined in the following manner: given a (not necessarily normed) state ρ the probability of obtaining 'no signal' is

$$\frac{\text{Tr}((1_a \times P_b^0)\rho)}{\text{Tr}(\rho)} \qquad \text{with } P_b^0 := |v\rangle\langle v| \tag{5}$$

and the density matrix of the sub-ensemble sorted out according to this effect

$$\text{Tr}_b((1_a \times P_b^0)\rho) \times |v\rangle\langle v| = (1_a \times P_b^0)\rho(1_a \times P_b^0). \tag{6}$$

The density matrix of those systems sorted according to 'signal obtained' on the other hand is set

$$\text{Tr}_b((1_a \times P_b^>)\rho) \times |v\rangle\langle v| \qquad \text{with } P_b^> := 1_b - P_b^0. \tag{7}$$

Remark. The quotient of the trace of the new density matrices and the trace of ρ (i.e. the probability (5) in the first case) denotes the part of the systems in the respectively sorted sub-ensembles.

Remark. The operation (6) is just the von Neumann–Lüders rule for the description of the sorted sub-ensemble and (7) suggests itself as a generalization if the radiation field is set back to the vacuum due to absorbing measurements.

The nth-order probabilities introduced in section 1 can now be computed, theoretically, by iterating all these operations specified according to the respective $\#_i$ and a final performance of the trace. Using the notation above their form becomes, however, rather involved. The probability of receiving a signal in only one prescribed interval is, for example, given by

$$W^{(1)}(t_1\Delta_1 > |t_0\rho_{a0}) = \text{Tr}[(1_a \times P_b^>)T(t_1 + \Delta_1, t_1)(\text{Tr}_b\, T(t_1, t_0)(\rho_{a0} \times |v\rangle\langle v|)) \times |v\rangle\langle v|]$$

and corresponding longer expressions would result for higher order probabilities. It is, therefore, useful to introduce the following time evolutions contracted on the atomic space.

(i) The unconditioned contracted evolution (Bloch dynamics):

$$\bar{T}_a(t, t')\rho_a := \text{Tr}_b(T(t, t')(\rho_a \times |v\rangle\langle v|)). \tag{8}$$

(ii) The (with respect to P_b^0) conditioned contracted evolution:

$$T_a^0(t, t')\rho_a := \text{Tr}_b((1_a \times P_b^0)T(t, t')(\rho_a \times |v\rangle\langle v|)). \tag{9}$$

(iii) The (with respect to $P_b^>$) conditioned contracted evolution:

$$T_a^>(t, t')\rho_a := \text{Tr}_b((1_a \times P_b^>)T(t, t')(\rho_a \times |v\rangle\langle v|)). \tag{10}$$

From these definitions it necessarily follows that

$$T_a^>(t, t') = \bar{T}_a(t, t') - T_a^0(t, t'). \tag{11}$$

By means of these operators the common probabilities (1), as calculated from quantum theory, can be written

$$W^{(n)}(t_n\Delta_n \#_n, \ldots, t_1\Delta_1 \#_1 | t_0\rho_{a0})$$

$$= \text{Tr}_a(T_a^{\#_n}(t_n + \Delta_n, t_n)\bar{T}_a(t_n, t_{n-1} + \Delta_{n-1}) \ldots T_a^{\#_1}(t_1 + \Delta_1, t_1)\bar{T}_a(t_1, t_0)\rho_{a0}). \tag{12}$$

In the following the question, whether a classical counting process can be found being adopted to these common probabilities in such a way that 'a signal' corresponds to at least one event during the interval and 'no-signal' to no event in the interval, will be investigated. The result (12) shows that this will be possible if, and only if, the contracted time evolution operators (8) and (9) satisfy the conditions

$$
\begin{aligned}
\bar{T}_a(t, t_0) &= \bar{T}_a(t, t_1)\bar{T}_a(t_1, t_0) \\
T_a^0(t, t_0) &= T_a^0(t, t_1)T_a^0(t_1, t_0)
\end{aligned}
\tag{13}
$$

and, following from those,

$$
T_a^>(t, t_0) = T_a^>(t, t_1)T_a^>(t_1, t_0) + T_a^>(t, t_1)T_a^0(t_1, t_0) + T_a^0(t, t_1)T_a^>(t_1, t_0).
$$

This would guarantee, as one can easily see, the compatibility conditions of the (calculated) common distributions (12), i.e.

$$
W(t_2\Delta_2 \#_2 | t_0, \rho_0) = \sum_{\#_1} W(t_2\Delta_2 \#_2, t_1\Delta_1 \#_1 | t_0, \rho_0)
$$

and respective relations for higher order common probabilities, which imply the existence of a point process.

The conditions (13) can never be fulfilled for a contracted dynamics. However, as demonstrated in [11] one has instead, for $T_a = \bar{T}_a$, T_a^0 and $t - t_1$, $t_1 - t_0$ exceeding a characteristic time span τ_m (the Markov time, see next section[†])

$$
T_a(t, t_0) = T_a(t, t_1)(1_a + E_a(t_1))T_a(t_1, t_0).
\tag{14}
$$

This allows the following important conclusion: if the 'piecing-effect operator' $E_a(t)$ is sufficiently small then there can certainly be found a classical counting process describing the common distributions for orders not too high (i.e. not too many measurements) sufficiently well. And, indeed, a natural way arises to construct such a process. For, as shown in section 3, the operators $\bar{T}_a$ and T_a^0 satisfy approximately the following initial-value problems:

$$
\frac{\mathrm{d}}{\mathrm{d}t} T_a^0(t, t')\rho_a = \mathscr{L}_a^0(t)T_a^0(t, t')\rho_a \qquad T_a^0(t, t)\rho_a = \rho_a
\tag{15a}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \bar{T}_a(t, t')\rho_a = \bar{\mathscr{L}}_a(t)\bar{T}_a(t, t')\rho_a \qquad \bar{T}_a(t, t)\rho_a = \rho_a.
\tag{15b}
$$

Here 'approximately' means:
(i) if $t - t' \gg \tau_m$;
(ii) up to a (multiplicative) initial error of the same order as the piecing-effect.

The solutions of the differential equations (15) have, by definition, the (semigroup) property (13) and thus determine the counting process which, for not too many observations and not too short measuring intervals, describes the idealized measurements introduced in section 1[‡]. Due to the compatibility condition it is determined already by

$$
W^{(n)}(t_n\Delta_n, \ldots, t_1\Delta_1 | t_0\rho_0) \coloneqq W^{(n)}(t_n\Delta_n >, \ldots, t_i\Delta_i >, \ldots, t_1\Delta_1 > | t_0\rho_0)
\tag{16}
$$

i.e. the probability that photons are registered during each of the measuring intervals.

[†] In the radiation problems considered here the Markov time is of the order of the (smallest) inverse transition frequency.
[‡] From the investigations in [11] it can be expected that the correlation time and the piecing-effect are, in the case of the radiative decay considered here, so small that all performable measurements can be described by the constructed process. This justifies our use of the term 'emitted photons'.

Further properties of the process can be derived only for restricted classes of models. For such investigations the following properties of the generators $\mathscr{L}_a^0$ and $\bar{\mathscr{L}}_a$ derived in section 3 turn out to be useful:

(i) $\bar{\mathscr{L}}_a(t)$ has the decomposition

$$\bar{\mathscr{L}}_a(t) =: \mathscr{L}_a^0(t) + \mathscr{L}_a^1. \tag{17}$$

(ii) The action of $\mathscr{L}_a^0$ can be written in the form

$$\mathscr{L}_a^0(t)\rho_a = -\mathrm{i}(\bar{H}_a(t)\rho_a - \rho_a\bar{H}_a^+(t)) \qquad \text{with } \bar{H}_a(t) := H_a(t) - \mathrm{i}D_a. \tag{18}$$

From (17) it follows immediately:

$$T_a^>(t, t')\rho_a = \int_{t'}^t \mathrm{d}s\, T_a^0(t, s)\mathscr{L}_a^1\bar{T}_a(s, t')\rho_a \tag{19}$$

$$\bar{T}_a(t, t') = T_a^0(t, t') + \int_{t'}^t \mathrm{d}s\, T_a^0(t, s)\mathscr{L}_a^1\bar{T}_a(s, t') \tag{20}$$

and a similar pair of equations with $\bar{T}_a$ and T_a^0 commuted in the integral.

This allows, for example, the following transformations of the probabilities (16):

$$W^{(1)}(t_1\Delta_1|t_0\rho_{a0}) = \int_{t_1}^{t_1+\Delta_1} \mathrm{d}s_1\, \mathrm{Tr}_a(T_a^0(t_1+\Delta_1, s_1)\mathscr{L}_a^1\bar{T}_a(s_1, t_0)\rho_{a0})$$

$$W^{(2)}(t_2\Delta_2, t_1\Delta_1|t_0\rho_{a0})$$

$$= \int_{t_2}^{t_2+\Delta_2} \mathrm{d}s_2 \int_{t_1}^{t_1+\Delta_1} \mathrm{d}s_1\, \mathrm{Tr}_a(\mathscr{L}_a^1 T_a^0(s_2, s_1)\mathscr{L}_a^1\bar{T}_a(s_1, t_0)\rho_{a0}) \tag{21}$$

$$+ \int_{t_2}^{t_2+\Delta_2} \mathrm{d}s_2 \int_{t_1+\Delta_1}^{t_2} \mathrm{d}s \int_{t_1}^{t_1+\Delta_1} \mathrm{d}s_1\, \mathrm{Tr}_a(T_a^0(t_2+\Delta_2, s_2)$$

$$\times \mathscr{L}_a^1\bar{T}_a(s_2, s)\mathscr{L}_a^1 T_a^0(s, s_1)\mathscr{L}_a^1\bar{T}_a(s_1, t_0)\rho_{a0}).$$

In the same way the higher probabilities may be expressed in similar terms. To calculate them requires, therefore, the solutions of the differential equations (15a) and (15b) and the determination of the superoperator $\mathscr{L}_a^1$.

As obvious from the representation (21) the processes become especially simple, if the models have the property that independently of ρ_a always

$$\mathscr{L}_a^1\rho_a = \rho_a^*\Lambda(\rho_a) \qquad \mathrm{Sp}_a(\rho_a^*) = 1 \tag{22}$$

holds (with a scalar function Λ and a fixed density matrix ρ_a^*). For then they are renewal processes with:

$$W_0(t, t') := \mathrm{Tr}_a(T_a^0(t, t')\rho_a^*)$$

being the probability that after an event in t' none will appear up to t;

$$W_1(t, t') := \Lambda(T_a^0(t, t')\rho_a^*)$$

being the probability density that after an event in t' the next will be registered in t;

$$\tilde{W}_1(t, t') := \Lambda(\bar{T}_a(t, t')\rho_a^*)$$

being the probability density that after an event in t' any will happen in t.

The last two densities are linked by the integral relation

$$\bar{W}_1(t, t') = W_1(t, t') + \int_{t'}^{t} \mathrm{d}s \, W_1(t, s) \bar{W}_1(s, t') \tag{23}$$

which holds for renewal processes and follows here from (20).

Concerning the class of models which fulfil (22) it is, therefore, not necessary to solve the Bloch equations (15*b*), but one can confine oneself on the system of differential equations (15*a*), or, due to (18), equivalently determine the effective time evolution-operator according to

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{U}(t, t') = -\mathrm{i}\bar{H}(t)\bar{U}(t, t'). \tag{24}$$

The relevance of this class of models will be demonstrated by the examples in section 4. (Simple generalizations of (22), e.g. as a sum of two terms, are obvious.)

3. Calculation of the operators $\mathscr{L}_a^0(t)$ and $\mathscr{L}_a^1$

As shown in the preceding section, the calculation of the photon statistics requires the knowledge of the generators $\mathscr{L}_a^0(t)$ (or $\bar{H}_a(t)$) and $\mathscr{L}_a^1$. To this end one has to construct linear differential equations approximately fulfilled by $T_a^0(t, t')\rho_a$ and $\bar{T}_a(t, t')\rho_a$. These can be obtained for both contracted time development operators in one calculation, setting

$$T_a(t, t')\rho_a := \mathrm{Tr}_b((1_a \times P_b)T(t, t')(\rho_a \times |v\rangle\langle v|)) \tag{25}$$

where P_b represents either P_b^0 or 1_b.

To derive the differential equation the projector formalism is especially suited, for it elucidates the nature of the approximations (Born–Markov) involved. The technical procedure has been presented extensively in [11]†. Therefore I shall confine myself mainly to the results and their discussion.

In a first step one can demonstrate that $T_a(t, t')$ satisfies exactly an integrodifferential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} T_a(t, t')\rho_a = -\mathrm{i}[H_a(t), T_a(t, t')\rho_a] - \int_0^{t-t'} \mathrm{d}s \, K_a(t, t-s) T_a(t-s, t')\rho_a. \tag{26}$$

The integral kernel is a complicated superoperator on the atomic space which, however, allows for $t - t'$ exceeding some characteristic time τ_m the substitution

$$\int_0^{t-t'} \mathrm{d}s \, K_a(t, t-s) T_a(t-s, t') = \int_0^{\infty} \mathrm{d}s \, K_a(t, t-s) T_{0a}(t-s, t) T_a(t, t'). \tag{27}$$

T_{0a} denotes the free atomic time evolution. Thus, for $t - t'$ sufficiently large, $T_a(t, t')$ satisfies a differential equation; and substituting the solution of (26) by that of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} T_a(t, t')\rho_a = \mathscr{L}_a(t) T_a(t, t')\rho_a \qquad T_a(t, t)\rho_a = \rho_a \tag{28}$$

† See also [12].

with

$$\mathscr{L}_a(t)\rho_a = -i[H_a(t),\rho_a] - G_a(t)\rho_a \tag{29}$$

$$G_a(t)\rho_a = \int_0^\infty ds\, K_a(t,t-s)T_{0a}(t-s,t)\rho_a \tag{30}$$

(i.e. taking $T_a(t,t')$ to satisfy the differential equation for all times) we get the Markov approximation. τ_m will therefore be called Markov time. $G_a(t)(K_a(\cdot,\cdot))$ can be approximately calculated (Born approximation) to yield

$$G_a(t)\rho_a = \int_0^\infty ds\, \mathrm{Tr}_b((1_a\times P_b)[H_{ab},T_0(t,t-s)[H_{ab},T_{0a}(t-s,t)\rho_a\times|v\rangle\langle v|]]). \tag{31}$$

Remark. Applying the Markov approximation results in an initial error caused by the deviation of the solutions of (26) and (28) during the initial interval ($t-t'<\tau_m$). It is plausible (see [11]) that an error of similar order will arise at each 'piecing' of the (total) time evolution in points $\bar{t}_\lambda$, i.e. by an experimentally performed operation described by a resetting of the density matrix according to $\rho(\bar{t}_\lambda)\rightarrow \mathrm{Tr}_b((1_a\times P_b)\rho(\bar{t}_\lambda))\times|v\rangle\langle v|$, for such an operation has no correspondence in the solution of (28). Therefore, the solution of the differential equation approximates best to the 'unpieced' time evolution or, in other words, the Markov approximation restricts the number of piecings. Thus approaches seem misleading which associate the solution of (28) in the case of $T_a^0(t,t')$ (where the piecing corresponds to a measurement and sorting out the no photon subensemble) to a dense series of 'gedanken measurements', although another derivation of (28)—here called the interval method and discussed in the appendix—may suggest such an interpretation. On the other hand the 'piecing-effect' (denoted by E_a in section 2) is so small that for all practically possible intervening measurements it can be neglected [11].

The generators $\mathscr{L}_a^0(t)$ and $\bar{\mathscr{L}}_a(t)$ are obtained by substituting P_b by P_b^0 or 1_b respectively. For the Hamilton operator as given by equation (2a) a straightforward calculation [11] yields†

$$\mathscr{L}_a^0(t)\rho_a = -i[H_a(t),\rho_a] - \sum_n |\kappa_n|^2\{C_n^+\tilde{C}_n(t)\rho_a + \rho_a\tilde{C}_n^+(t)C_n\} \tag{32a}$$

$$\bar{\mathscr{L}}_a(t)\rho_a = \mathscr{L}_a^0(t)\rho_a + \sum_n |\kappa_n|^2\{\tilde{C}_n(t)\rho_a C_n^+ + C_n\rho_a\tilde{C}_n^+(t)\} \tag{32b}$$

with‡

$$\tilde{C}_n(t) = \int_0^\infty ds\, e^{-i\omega_n s}T_{0a}(t,t-s)C_n. \tag{33}$$

As obvious by equation (32a), the action of the generator $\mathscr{L}_a^0(\cdot)$ can be expressed in the form

$$\mathscr{L}_a^0(t)\rho_a = -i\{\bar{H}_a(t)\rho_a - \rho_a\bar{H}_a^+(t)\} \tag{34}$$

where (time independence of $\tilde{C}_n$ assumed)

$$\bar{H}_a(t) := H_a(t) - iD_a := H_a(t) - i\sum_n |\kappa_n|^2 C_n^+\tilde{C}_n \tag{35}$$

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† During these calculations it becomes obvious that the Markov time is of the order of the inverse of the (smallest) transition frequency considered.

‡ Since $\|V_a(\cdot)\|\tau_m \ll 1$ one can in general neglect $V_a(\cdot)$ in the calculation of $\tilde{C}_n(\cdot)$. This operator becomes therefore time independent. This approximation is made throughout in the following section.

is sometimes called the effective Hamilton operator. This yields for $T_a^0(\cdot,\cdot)$ the representation

$$T_a^0(t, t')\rho_a = \bar{U}_a(t, t')\rho_a \bar{U}_a^+(t, t') \tag{36}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{U}_a(t, t') = -\mathrm{i}\bar{H}_a(t)\bar{U}_a(t, t') \qquad \bar{U}_a(t, t) = 1.$$

4. Illustration by simple examples

In this section the practical calculation of $\bar{H}_a(t)$ and $\mathscr{L}_a^1$ shall be demonstrated by means of four specific models similar to those discussed in the literature to explain performed experiments. With the exception of the last all of them belong to the class, characterized by condition (22). ($\bar{U}_a$ could be also calculated under suitable approximations.)

Model A. The laser-driven two-level system.
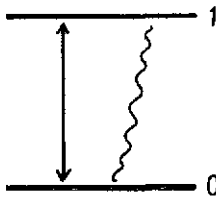
In this case the Hamilton operator (rotating wave approximation) reads

$$H(t) = \omega_0|1\rangle\langle1| + \tfrac{1}{2}\Omega(|1\rangle\langle0| \, \mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{HC}) + \sum_n \omega_n a_n^+ a_n + \sum_n (\kappa_n|0\rangle\langle1|a_n^+ + \mathrm{HC}). \tag{37}$$

Therefore

$$H_a(t) = \omega_0|1\rangle\langle1| + \tfrac{1}{2}\Omega(|1\rangle\langle0| \, \mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{HC}) = H_a + V_a(t)$$
$$C_n = |0\rangle\langle1| =: C.$$

From (23) it follows (neglecting $V_a(t)$) that:

$$\tilde{C}_n = \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-\mathrm{i}\omega_n s}(\mathrm{e}^{-\mathrm{i}\omega_0 s}|1\rangle\langle1| + |0\rangle\langle0|)|0\rangle\langle1|(\mathrm{e}^{\mathrm{i}\omega_0 s}|1\rangle\langle1| + |0\rangle\langle0|)$$

$$= \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-\mathrm{i}(\omega_n - \omega_0)s}C$$

$$= \left[\mathscr{P}\frac{-\mathrm{i}}{\omega_n - \omega_0} + \pi\delta(\omega_n - \omega_0)\right]C$$

$$=: f(\omega_n, \omega_0)C.$$

Hence, in the continuum limit $\{\kappa_n\} \to \kappa(\cdot)$, one gets

$$\bar{H}_a(t) = H_a(t) - \mathrm{i}\int_0^\infty \mathrm{d}\omega|\kappa(\omega)|^2 f(\omega, \omega_0)C^+C$$

$$= H_a(t) + \left\{-\mathscr{P}\int_0^\infty \mathrm{d}\omega\frac{|\kappa(\omega)|^2}{\omega - \omega_0} - \mathrm{i}\pi|\kappa(\omega_0)|^2\right\}|1\rangle\langle1|.$$

By neglecting, as usually justified in quantum optics, the principal value integral and with the abbreviation

$$\Gamma := \pi |\kappa(\omega_0)|^2$$

one obtains finally

$$\bar{H}_a(t) = (\omega_0 - i\Gamma)|1\rangle\langle 1| + \tfrac{1}{2}\Omega(e^{-i\omega_L t}|1\rangle\langle 0| + \text{HC}). \tag{38}$$
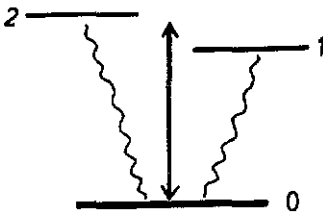
Similar calculations yield

$$\mathscr{L}_a^1 \rho_a = \int_0^\infty d\omega |\kappa(\omega)|^2 \{f(\omega, \omega_0) + f^*(\omega, \omega_0)\} C \rho_a C^+$$

$$= 2\Gamma|0\rangle\langle 1|\rho_a|1\rangle\langle 0|$$

$$= |0\rangle\langle 0|2\Gamma\rho_{a11}. \tag{39}$$

The result is of the form (22) with $\rho_a^* = |0\rangle\langle 0|$ and $\Lambda(\rho_a) = 2\Gamma\rho_{a11}$.

Concerning the following two models details of the calculations, which are similar, are omitted, and only the results are written down.

Model B. The three-level V-system driven by a single laser.

Here the Hamilton operator reads

$$H(t) = \omega_2|2\rangle\langle 2| + \omega_1|1\rangle\langle 1| + \tfrac{1}{2}[e^{i\omega_L t}\{\Omega_2|0\rangle\langle 2| + \Omega_1|0\rangle\langle 1|\} + \text{HC}]$$

$$+ \sum_n \omega_n a_n^+ a_n + \sum_n [\kappa_n\{\lambda_{2n}|0\rangle\langle 2| + \lambda_{1n}|0\rangle\langle 0|\}a_n^+ + \text{HC}]$$

and therefore

$$C_n = \lambda_{2n}|0\rangle\langle 2| + \lambda_{1n}|0\rangle\langle 1|.$$

Similar approximations as in model A yield the effective Hamilton operator

$$\bar{H}_a(t) = H_a(t) - i\{\Gamma_{22}|2\rangle\langle 2| + \Gamma_{11}|1\rangle\langle 1| + \Gamma_{21}|2\rangle\langle 1| + \Gamma_{12}|1\rangle\langle 2|\} \tag{40}$$
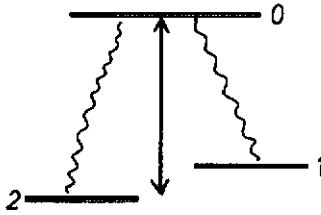
where

$$\Gamma_{22} = \pi|\lambda_2(\omega_2)\kappa(\omega_2)|^2 \qquad \Gamma_{11} = \pi|\lambda_1(\omega_1)\kappa(\omega_1)|^2$$

$$\Gamma_{21} = \pi\lambda_2^*(\omega_1)\lambda_1(\omega_1)|\kappa(\omega_1)|^2 \qquad \Gamma_{12} = \pi\lambda_1^*(\omega_2)\lambda_2(\omega_2)|\kappa(\omega_2)|^2.$$

Using the same notation one gets

$$\mathscr{L}_a^1 \rho_a = |0\rangle\langle 0|\{2\Gamma_{22}\rho_{a22} + 2\Gamma_{11}\rho_{a11} + [\Gamma_{21} + \Gamma_{12}^*]\rho_{a12} + [\Gamma_{12} + \Gamma_{21}^*]\rho_{a21}\}. \tag{41}$$

Model C. The three-level Λ-system driven by a single laser.

In this case the Hamilton operator is

$$H(t) = -\omega_2|2\rangle\langle 2| - \omega_1|1\rangle\langle 1| + \tfrac{1}{2}[e^{i\omega_L t}\{\Omega_2|2\rangle\langle 0| + \Omega_1|1\rangle\langle 0|\} + \text{HC}]$$
$$+ \sum_n \omega_n a_n^+ a_n + \sum_n [\kappa_n\{\lambda_{2n}|2\rangle\langle 0| + \lambda_{1n}|1\rangle\langle 0|\}a_n^+ + \text{HC}]$$

and thus obviously

$$C_n = \lambda_{2n}|2\rangle\langle 0| + \lambda_{1n}|1\rangle\langle 0|.$$

Neglecting again the principle value integral, one finds

$$\bar{H}_a(t) = H_a(t) - i\{\Gamma_{22} + \Gamma_{11}\}|0\rangle\langle 0| \tag{42}$$

with Γ_{22}, Γ_{11} defined as in model B. If, in addition, the abbreviation

$$2\tilde{\Gamma} = \pi\lambda_2(\omega_1)\lambda_1^*(\omega_1)|\kappa(\omega_1)|^2 + \pi\lambda_1^*(\omega_2)\lambda_2(\omega_2)|\kappa(\omega_2)|^2$$

is used, one obtains, written in the form (22),

$$\mathcal{L}_a^1\rho_a = \frac{1}{\Gamma_{22}+\Gamma_{11}}\{\Gamma_{22}|2\rangle\langle 2| + \Gamma_{11}|1\rangle\langle 1| + \tilde{\Gamma}|2\rangle\langle 1| + \tilde{\Gamma}^*|1\rangle\langle 2|\}2(\Gamma_{22}+\Gamma_{11})\rho_{a00}. \tag{43}$$

Model D. As model A without rotating-wave approximation. This is the most simple model which does not belong to the class (22) and it is obtained by setting $|0\rangle\langle 1| + |1\rangle\langle 0|$, instead of $|0\rangle\langle 1|$ in model A, for the operator C. Evaluating as above one finds $\mathcal{L}_a^0(\bar{H}_a)$ to be unchanged, but $\mathcal{L}_a^1$ is now given by

$$\mathcal{L}_a^1\rho_a = |0\rangle\langle 0|2\Gamma\rho_{a11} + |1\rangle\langle 0|\Gamma\rho_{a01} + |0\rangle\langle 1|\Gamma\rho_{a10}. \tag{44}$$

The expression on the right seems at first sight questionable, since it is not positive. However, in $T_a^>(t, t')\rho_a$ as defined by (19) the contribution of the two last terms vanishes for $t - t' \gg \tau_m \approx 1/\omega_0$. Therefore the result coincides with that for model A for measurement intervals in the allowed regime.

Acknowledgments

Appendix

Comparison with the interval method

The investigations of section 2 were essentially based on the (conditioned) time-evolution $T_a(t, t')\rho_a$, and in section 3 a differential equation for it could be obtained

by means of the projector formalism. In principle one can consider another kind of time evolution $T_{a\Delta}(t, t')\rho_a$ constructed by successive multiplication of operators $T_a(\cdot, \cdot)$ calculated for small time increments Δ. 'Pieced' contracted time-evolution operators like this have often been applied in the theory of heat baths [13], and recently were used to investigate photon-statistics problems [8, 9]. Throughout, however, explicit calculations require perturbation theory (Born approximation) and the Markov approximation. In this appendix it is shown that such a procedure, called 'interval method' here, yields a time evolution (approximately) compatible with $T_a(t, t')\rho_a$ while the effects of the piecings are completely washed out.

One starts by calculating

$$T_a(t+\Delta, t)\rho_a = \mathrm{Tr}_b((1_a \times P_b)T(t+\Delta, t)(\rho_a \times |v\rangle\langle v|)). \tag{A1}$$

In the interaction picture the time evolution (of the composed system, atom + field) has the decomposition

$$T(t+\Delta, t)\rho = T_0(t+\Delta, t)\tilde{T}(t+\Delta, t)\rho \tag{A2}$$

with

$$\frac{d}{ds}\tilde{T}(t+s, t)\rho = -i[(T_0(t, t+s)H_{ab}), \tilde{T}(t+s, t)\rho] \qquad \tilde{T}(t, t)\rho = \rho. \tag{A3}$$

Second-order perturbation calculation yields for the solution of the initial value problem (A3)

$$\tilde{T}(t+\Delta, t)\rho \approx \rho - i \int_0^\Delta ds[(T_0(t, t+s)H_{ab}), \rho]$$

$$- \int_0^\Delta ds \int_0^s ds'[(T_0(t, t+s)H_{ab}), [(T_0(t, t+s')H_{ab}), \rho]]. \tag{A4}$$

If the expression (A4) with $\rho = \rho_a \times |v\rangle\langle v|$ is substituted into equation (A2) and this itself into (A1), then a short calculation yields

$$T_a(t+\Delta, t)\rho_a$$

$$\approx T_{0a}(t+\Delta, t)\left\{\rho_a - \int_0^\Delta ds \int_0^s ds'\, T_{0a}(t, t+s)\right.$$

$$\times \mathrm{Tr}_b((1_a \times P_b)[H_{ab}, T_0(t+s, t+s-s')$$

$$\left.\times [H_{ab}, T_{0a}(t+s-s', t)\rho_a \times |v\rangle\langle v|]])\right\}. \tag{A5}$$

By performing the Markov approximation (inner integration boundary to infinity assuming $\Delta \gg \tau_m$) this simplifies to

$$T_a(t+\Delta, t)\rho_a = T_{0a}(t+\Delta, t)\left\{\rho_a - \int_0^\Delta ds\, T_{0a}(t, t+s)G_a(t+s)T_{0a}(t+s, t)\rho_a\right\}. \tag{A6}$$

Written differentially it reads

$$\frac{\partial}{\partial\Delta}T(t+\Delta, t)\rho_a = -i[H_a(t+\Delta), T_a(t+\Delta, t)\rho_a] - G_a(t+\Delta)T_{0a}(t+\Delta, t)\rho_a. \tag{A7}$$

This shows that the result (A6) may be used either to approximate (iteratively) the solution of (28)-(30) for time increments satisfying $\Delta \| G_a \| \ll 1$ [9] or to determine a differential equation for the contracted time evolution [8, 13] which, however, is identical with (28). The reason is that due to the Markov approximation the effect of the piecing (fictitious measurements) is systematically suppressed. With respect to the approaches as [8, 9] one has to conclude that the gedanken measurements introduced are effectively not performed.

References

[1] Dehmelt M G 1974 *Bull. Am. Phys. Soc.* **20** 60
[2] Glauber R J 1963 *Phys. Rev.* **130** 2529; 1963 *Phys. Rev.* **131** 2766
   Kelly P L and Kleiner W H 1964 *Phys. Rev.* **136** A316
[3] Javanainen J 1985 *Phys. Rev.* A **33** 2121
[4] Schenzle A, DeVoe R G and Brewer R G 1986 *Phys. Rev.* A **33** 2127
[5] Lüders G 1951 *Ann. Phys., Lpz.* **8** 323
[6] Zoller P, Marte M and Walls D 1986 *Phys. Rev.* A **35** 198
[7] Porrati M and Putterman S 1987 *Phys. Rev.* A **36** 929; 1988 *Phys. Rev.* A **39** 3010
[8] Hegerfeldt G C and Wilser T S 1991 *Proc. 2nd Wigner Symposium (Goslar, 1991)* ed H D Doebner, W Scherer and F Schroeck (Singapore: World Scientific)
   Wilser T 1991 *Dissertation* Göttingen
   Hegerfeldt G C 1992 *Phys. Rev. A* How to reset an atom after a photon emission: Applications to photon counting processes *Preprint* Göttingen
[9] Dalibard J, Castin Y and Mölmer K 1992 *Phys. Rev. Lett.* **68** 581
[10] Davies E B 1976 *Quantum Theory of Open Systems* (New York: Academic)
   Srinivas M D and Davies E B 1981 *Opt. Acta* **28** 981
[11] Reibold R 1992 *Physica A* On the 'piecing-effect' of the projector formalism and its application on repeated measurements in quantum optics *Preprint* Göttingen
[12] Haake F 1973 *Springer Tracts in Modern Physics* (Quantum Statistics) **66** (Berlin: Springer) p 98
   Agarwal G S 1974 *Springer Tracts in Modern Physics* (Quantum Optics) **70** (Berlin: Springer)
   Fick E and Sauermann G 1983 *Quantenstatistik Dynamischer Prozesse* Bd I (Frankfurt: Thun Deutsch)
[13] Louisell W G 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)